On the Construction of Relativistic Representation Spaces in Functional Quantum Theory of the Nonlinear Spinor Field

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Functional quantum theory is defined by an isomorphism of the state space H of a conventional quantum theory into an appropriate functional state space $\mathfrak{H}$. It is a constructive approach to quantum theory in those cases where the state spaces H of physical eigenstates cannot be calculated explicitly like in nonlinear spinor field quantum theory. For the foundation of functional quantum theory appropriate functional state spaces have to be constructed which have to be representation spaces of the corresponding invariance groups. In this paper, this problem is treated for the spinor field. Using anticommuting source operator, it is shown that the construction problem of these spaces is tightly connected with the construction of appropriate relativistic function spaces. This is discussed in detail and explicit representations of the function spaces are given. Imposing no artificial restrictions it follows that the resulting functional spaces are indefinite. Physically the indefiniteness results from the inclusion of tachyon states. It is reasonable to assume a tight connection of these tachyon states with the ghost states introduced by Heisenberg for the regularization of the nonrenormalizable spinor theory.

Functional quantum theory is defined by an isomorphism of the state space H of a conventional quantum theory into an appropriate functional state space $\mathfrak{H}$. It can be considered to be constructive approach to quantum theory in those cases where the state spaces H of physical eigenstates cannot be calculated explicitly like in nontrivial relativistic quantum field theories. Additionally it provides a framework for the physical and mathematical foundation of functionals which have been used so far only formally in quantum field theory. It has been introduced by Stumpf [1, 2] and applied especially to the problem of the definition and calculation of global observables in nonlinear spinor field quantum theory [3] of Heisenberg and coworkers [4, 5]. For the map of a spinor quantum field theory spinorial functional state spaces $\mathfrak{H}$ are required. As we are interested only in this type of theories we consider spinorial functional spaces only. In order to satisfy the conditions for a map of H into $\mathfrak{H}$, the spinorial functional state spaces have to be representation spaces of the corresponding symmetry groups. In this case it is the Poincaré group which we shall consider only, as the treatment of additional symmetry groups is an evident generalization. A first treatment of such spaces has been given by Stumpf [6]. In this paper we give a more elaborate version of the approach used in [6]. Its main idea is to construct by means of a Jordan Wigner algebra generalized Fock spaces which exhibit the required transformation properties and can be identified with base representations of the required spinorial functional spaces. Functional spaces generated by Jordan Wigner algebras have been already studied by many authors, compare e.g. Berezin [7], Coester [8]. But it will be shown in the following that the connection of a Jordan Wigner algebra with appropriate transformation properties leading to the required spinorial functional base representations is by no means trivial and requires a thorough investigation which so far has not yet been done. Working on different lines, Rzewuski et al. have undertaken also investigations of functional spaces with considerable perfection [21]. They treat the problem of anticommuting sources by a mapping of conventional function spaces of commuting sources by means of an ordering sign function, defining thereby an isomorphism between the two types of spaces. But in contrary to the program persued here, functional spaces for conventional coupling theories with fixed masses are considered, while in our formulation due to the general concept of nonlinear spinor theory no special masses are distinguished. Thus the approach made here is more general, as will also be seen from the results. Finally it should be noted that for Boson functional state spaces Rieckers [22] has made a thorough investigation about the connection of function-spaces and functional spaces which in principle runs on the same pattern as the spinorial problem treated in [6] and in this paper.

1. Fundamentals

To provide a suitable basis for our investigation we give a short review about the propositions which have been discussed already in detail in preceding papers [3, 6]. We denote by H the ordinary Hilbert space of nonlinear spinor theory and by $|a\rangle \in H$ an eigenstate of this theory with quantum numbers a, while $|0\rangle$ is the physical groundstate of the theory. Omitting for simplicity the isospin dependence, the spinor field is defined by a Hermitean spinor field operator $\Psi_\alpha(x)$ with $\alpha = (\varrho, \beta)$. To give this Hermitean field a definite meaning we connect it with an ordinary spinor field $\psi_\beta(x)$ by the definitions

$$\psi_\beta(x) = : \varphi_{1\beta}(x)\,; \quad \psi_\beta^+(x) = : \varphi_{2\beta}(x) \tag{1.1}$$

and

$$\Psi_{\varrho\beta}(x) := C_{\varrho\lambda}^{-1} \varphi_{\lambda\beta}(x) \tag{1.2}$$

with

$$C_{\varrho\lambda}^{-1} := 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \tag{1.3}$$

Performing a Poincaré transformation

$$x_\mu' = \Lambda_\mu^{\ \nu} x_\mu + d_\mu \tag{1.4}$$

we assume H to be a representation space of this group. Then (1.4) generates a representation $U(\Lambda, d)$ and the eigenstates $|a\rangle$ undergo the transformation

$$|a\rangle' = U(\Lambda, d)\,|a\rangle \tag{1.5}$$

with $|0\rangle' \equiv |0\rangle$ while the ordinary spinor field $\psi_\beta(x)$ is assumed to transform like

$$U^{-1}(\Lambda, d)\,\psi_\beta(x')\,U(\Lambda, d) = S_{\beta\varkappa}(\Lambda)\,\psi_\varkappa\left(\Lambda^{-1}(x'-d)\right) \tag{1.6}$$

where $S_{\beta\varkappa}(\Lambda)$ is the transformation matrix of "classical" Dirac spinors. For the Hermitean spinor field (1.6) leads to

$$U^{-1}(\Lambda, d)\,\Psi_{\varrho\beta}(x')\,U(\Lambda, d) = D_{\varrho\beta}^{\lambda\varkappa}(\Lambda)\,\Psi_{\lambda\varkappa}\left(\Lambda^{-1}(x'-d)\right) \tag{1.7}$$

with

$$D_{\varrho\beta}^{\lambda\varkappa} = \begin{pmatrix} \operatorname{Re} S_{\beta\varkappa} & -\operatorname{Im} S_{\beta\varkappa} \\ \operatorname{Im} S_{\beta\varkappa} & \operatorname{Re} S_{\beta\varkappa} \end{pmatrix}. \tag{1.8}$$

It is interesting to note that $S(\Lambda)$ as well as $D(\Lambda)$ does not depend on any mass of a particle but only on the ordinary homogeneous transformation matrix Λ. Therefore the relation (1.6) as well as relation (1.7) is not restricted to spinor fields connected with a definite mass, but holds quite generally for an arbitrary spinor field as it is required in nonlinear spinor theory. Further the position of indices in (1.8) has a definite meaning, too. In analogy to vector and tensor transformation laws invariants can be constructed also with spinor indices [6].

Concerning the map into functional space we consider the quantities

$$\tau_n'(\overset{\alpha_1}{x_1} \ldots \overset{\alpha_n}{x_n} | a) := \langle 0 | T\, \Psi_{\alpha_1}(x_1) \ldots \Psi_{\alpha_n}(x_n) | a \rangle . \tag{1.9}$$

These quantities transform under a Poincaré transformation (1.4) like

$$\tau_n'(\overset{\alpha_1'}{x_1'} \ldots \overset{\alpha_n'}{x_n'} | a') = D_{\alpha_1'}^{\alpha_1}(\Lambda) \ldots D_{\alpha_n'}^{\alpha_n}(\Lambda)\, \tau_n'(\underset{\alpha_1}{x_1} \ldots \underset{\alpha_n}{x_n} | a). \tag{1.10}$$

They are the basic quantities to define state functionals. We introduce them by

$$|\mathfrak{T}(j,a)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \tag{1.11}$$

$$\cdot \int \tau_n(\underset{\alpha_1}{x_1} \ldots \underset{\alpha_n}{x_n} | a)\, | D_n(\overset{\alpha_1}{x_1} \ldots \overset{\alpha_n}{x_n}) \rangle \, \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n$$

where the set of functional base states $\{|D_n\rangle\}$ represents an appropriate functional space $\mathfrak{H}$ which has to be a representation space of the Poincaré group, too. To achieve this we introduce formally Hermitean spinorial source operators $j_\alpha(x)$ and $\partial_\alpha(x)$ satisfying the anticommutation relations

$$[j_\alpha(x), j_{\alpha'}(x')]_+ = [\partial_\alpha(x), \partial_{\alpha'}(x')]_+ = 0 \tag{1.12}$$

and

$$[j_\alpha(x), \partial^{\alpha'}(x')]_+ = \delta_\alpha^{\alpha'}\,\delta(x-x') \tag{1.13}$$

where the raising and lowering of indices is defined by

$$\partial^\alpha(x) = g^{\alpha\beta}\,\partial_\beta(x)\,; \quad j^\alpha(x) = g^{\alpha\beta}\,j_\beta(x) \tag{1.14}$$

with the metrical fundamental tensor $g^{\alpha\beta}$ in Hermitean spinor space [6]. To obtain the required transformation properties in $\mathfrak{H}$, we assume further the relations

$$\begin{aligned} V^{-1}(\Lambda, d)\,j_\alpha(x')\,V(\Lambda, d) &= D_\alpha^{\ \beta}(\Lambda)\,j_\beta\left(\Lambda^{-1}(x'-d)\right), \\ V^{-1}(\Lambda, d)\,\partial^\alpha(x')\,V(\Lambda, d) &= D_\beta^{-1\alpha}(\Lambda)\,\partial^\beta\left(\Lambda^{-1}(x'-d)\right). \end{aligned} \tag{1.15}$$

By these operators a functional Fock space can be created if one requires for the functional groundstate

$$\partial_\alpha(x)\,|\varphi_0\rangle = 0 . \tag{1.16}$$

As has been noticed in preceding papers this condition is not the only one which is possible [2, 6]. Rather an entire set of different groundstate definitions is admitted. The physical relevance of such different groundstate definitions has been emphasized by Rieckers [9], who also discussed an algebraic classification of these representations. In the following we restrict ourselves to the Fockspace representation defined by (1.16) and shall study its realization.

Proceeding formally one may define the following base functionals

$$|D_n(\overset{}{\underset{\alpha_1}{x_1}}\dots\underset{\alpha_n}{x_n})\rangle := \frac{1}{n!}\, j_{\alpha_1}(x_1)\dots j_{\alpha_n}(x_n)\,|\varphi_0\rangle \quad (1.17)$$

while the dual set in this space is given by the base functionals

$$\langle D^n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n})| := \langle\varphi_0|\,\partial^{\alpha_1}(x_1)\dots\partial^{\alpha_n}(x_n)\,1/n!\,. \quad (1.18)$$

Applying all the rules which have been given, one may derive then the functional scalar product

$$\langle D^n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n})|D_m(\underset{\alpha_1}{x_1'}\dots\underset{\alpha_n}{x_m'})\rangle = \delta^n_m \sum_{p\,\lambda_1\dots\lambda_n}(-1)^p\,\delta^{\alpha_1}_{\alpha'\lambda_1}\,\delta(x_1-x'_{\lambda_1})\dots\delta^{\alpha_n}_{\alpha'\lambda_n}\,\delta(x_n-x'_{\lambda_n})\,(1/n!)^2\,. \quad (1.19)$$

Therefore the set (1.17) defines a functional space $\mathfrak{H}$ equipped with the scalar product (1.19) between the original set and the dual set. The transformation properties of the original set (1.17) follow immediatly from (1.15)

$$V^{-1}(\Lambda,d)\,|D_n(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n})\rangle = D^{\alpha_1'}_{\alpha_1}(\Lambda)\dots D^{\alpha_n'}_{\alpha_n}(\Lambda)\,|D_n(\underset{\alpha_1}{\Lambda^{-1}(x_1-d)}\dots\underset{\alpha_n}{\Lambda^{-1}(x_n-d)})\rangle \quad (1.20)$$

if one assumes the Poincaré invariance of the functional groundstate i.e. $V(\Lambda,d)|\varphi_0\rangle \equiv |\varphi_0\rangle$. Combining this with the transformation property (1.10) one obtains

$$|\mathfrak{T}(j,a')\rangle' = V(\Lambda,d)\,|\mathfrak{T}(j,a)\rangle \quad (1.21)$$

which is the map of (1.5) in functional space.

In the following we are not interested in the further investigation of this map. Rather we investigate a realization of the quantities and relations (1.12) to (1.20) in order to secure their existence. Only having provided such an explicit realization one is allowed to operate further with quantities in functional space.

2. Base Function Expansions

To realize the functional operators introduced in the preceding section we use suitable base function expansions. We assume to have a complete set of linear independent real spinorial testfunctions in Minkowski space $\{f_{\alpha k}(x),\ 1 \leqq k < \infty\}$ having proper transformation properties unter Poincaré transformations. Such sets will be constructed explicitly in the next section. Imposing no artificial constraints it will be shown there that the spaces spanned up by such sets are indefinite. Like in ordinary Minkowski space it is convenient to introduce also in the corresponding function spaces dual base sets which we denote by $\{g_{\alpha k}(x),\ 1 \leqq k < \infty\}$. By definition for these sets the orthonormality relations

$$\int g_{\alpha k}(x)\,f^{\alpha}_{k'}(x)\,\mathrm{d}^4x = \delta^k_{k'} \quad (2.1)$$

hold and the completeness relation

$$\sum_k g_{\alpha k}(x)\,f^{\beta}_k(x') = \delta^{\beta}_{\alpha}\,\delta(x-x') \quad (2.2)$$

can be derived. Further we assume the transformation property

$$f'_{\alpha' k}(x') = D^{\alpha}_{\alpha'}(\Lambda)\,f_{\alpha k}(\Lambda^{-1}(x'-d)) = \sum_r C_{kr}(\Lambda,d)\,f_{\alpha' r}(x') \quad (2.3)$$

to be satisfied for the original set as well as for the dual set with a corresponding transformation matrix $\tilde{C}_{kr}(\Lambda,d)$. Therefore by this assumption these systems provide suitable representation spaces for the Poincaré group.

For the expansion of the functional operators we assume

$$j_\alpha(x) = \sum_k a_k^+\,f_{\alpha k}(x), \quad \partial_\alpha(x) = \sum_k a_k\,g_{\alpha k}(x) \quad (2.4)$$

to be valid. Then for the expansion coefficients $\{a_k, a_k^+,\ 1 \leqq k < \infty\}$ the statement holds

Stat. 2.1: If $j_\alpha(x)$ and $\partial_\alpha(x)$ satisfy (1.12) (1.13) and if (2.1) is assumed to be valid, then the set $\{a_k, a_k^+,\ 1 \leqq k < \infty\}$ is given by the elements of a Jordan Wigner algebra with the anticommutation relations

$$[a_k^+, a_l^+]_+ = [a_k, a_l]_+ = 0 \quad (2.5)$$

and

$$[a_k^+, a_l]_+ = \delta_{kl}\,. \quad (2.6)$$

Proof: From (1.12) and (2.4) follows due to the proposition

$$[j_a(x), j_{a'}(x')]_+ = \sum_{k k'} f_{ak}(x) f_{a'k'}(x') [a_k^+, a_{k'}^+]_+ = 0 . \tag{2.7}$$

Multiplying with $g_l^\alpha(x)$ from the left and $g_{l'}^{\alpha'}(x)$ from the right and integrating over x and x' with (2.1) one obtains (2.5). Further from (2.3) the relation

$$[j_a(x), \partial^\beta(x')]_+ = \sum_{k l} f_{ak}(x)\, g_l^\beta(x') [a_k^+, a_l]_+ = \delta_a^\beta\, \delta(x-x') \tag{2.8}$$

holds. Multiplying with $g_r^\alpha(x)$ from the left and $f_{\beta h}(x')$ from the right and integrating over x and x' with (2.1) one obtains (2.5), q. e. d.

Applying (2.4) to the base functionals (1.17) (1.18) the representation

$$| D_n(\overset{\alpha_1}{x_1} \dots \overset{\alpha_n}{x_n}) \rangle = \sum_{k_1 \dots k_n} f_{\alpha_1 k_1}(x_1) \dots f_{\alpha_n k_n}(x_n) | D_n(k_1 \dots k_n) \rangle \tag{2.9}$$

with

$$|D_n(k_1 \dots k_n) \rangle := \frac{1}{n!} a_{k_1}^+ \dots a_{k_n}^+ | \varphi_0 \rangle \tag{2.10}$$

and

$$\langle D^n(\overset{\alpha_n}{x_1} \dots \overset{\alpha_1}{x_n}) | = \sum_{k_1 \dots k_n} g_{k_1}^{\alpha_1}(x_1) \dots g_{k_n}^{\alpha_n}(x_n) \langle D^n(k_1 \dots k_n)| \tag{2.11}$$

with

$$\langle D^n(k_1 \dots k_n) | := \frac{1}{n!} a_{k_1} \dots a_{k_n} | \varphi_0 \rangle \tag{2.12}$$

results, while (1.16) is transformed into

$$a_k | \varphi_0 \rangle = 0 \quad \forall k . \tag{2.13}$$

Also (1.14) is satisfied by construction. By direct calculation we obtain from (2.5) (2.6) (2.13)

$$\langle D^n(k_1 \dots k_n) | D_m(k_1' \dots k_m') \rangle = \delta_n^m\, p \sum_{\lambda_1 \dots \lambda_n} (-1)^p\, \delta_{k_1 k'_{\lambda_1}} \dots \delta_{k_n k'_{\lambda_n}} (n!)^{-2} \tag{2.14}$$

which is in accordance with (1.19).

Finally we look for the transformation properties. There he have the

Stat. 2.2: If $j_a(x)$ and $\partial_a(x)$ satisfy (1.15) and if (2.3) is assumed to be valid, then the relations

$$\begin{aligned} V^{-1}(\Lambda, d)\, a_m^+ V(\Lambda, d) &= \sum_k C_{km}(\Lambda, d)\, a_k^+, \\ V^{-1}(\Lambda, d)\, a_m V(\Lambda, d) &= \sum_k \tilde{C}_{km}(\Lambda, d)\, a_k \end{aligned} \tag{2.15}$$

hold.

Proof: We consider only the transformation properties of $j_a(x)$, resulting in the transformation properties of a_k^+. The derivation of the corresponding transformation properties of a_k runs on the same pattern. By substitution of (2.4) into (1.15), the left hand side of (1.15) gives

$$V^{-1}(\Lambda, d)\, j_a(x') V(\Lambda, d) = \sum_k V^{-1}(\Lambda, d)\, a_k^+ f_{ak}(x') V(\Lambda, d) \tag{2.16}$$

while the right hand side gives due to (2.3)

$$\begin{aligned} D_a^\beta(\Lambda)\, j_\beta(\Lambda^{-1}(x'-d)) &= D_a^\beta(\Lambda) \sum_k a_k^+ f_{\beta k}(\Lambda^{-1}(x'-\mathrm{d})) \\ &= D_a^\beta(\Lambda) \sum_k a_k^+ D_\beta^{-1\gamma}(\Lambda) \sum_l C_{kl}(\Lambda, d) f_{\gamma l}(x') \\ &= \sum_{k l} a_k^+ C_{kl}(\Lambda, d) f_{al}(x') . \end{aligned} \tag{2.17}$$

Thus by (1.15)

$$\sum_k V^{-1}(\Lambda, d)\, a_k^+ f_{ak}(x') V(\Lambda, d) = \sum_{k l} a_k^+ C_{kl}(\Lambda, d) f_{al}(x') \tag{2.18}$$

holds. Multiplying with $g_j^\alpha(x')$ and integrating over x' the first formula of (2.15) results. The second formula is derived analogously, q. e. d.

Concerning the transformation properties (2.15) required for the operators a_k^+, a_k, $1 \leqq k < \infty$, we have the

Stat. 2.3: If the expansion system $f_{ak}(x)$ satisfies (2.1) and (2.3) then the conditions (2.15) are fulfilled identically without imposing any restriction to the operators a_k^+, a_k.

Proof: As any Poincaré transformation can be generated by infinitesimal ones, it suffices to consider a general infinitesimal Poincaré transformation. The generator of it in functional space we denote by $(1 + i\, \delta V)$. Then it was shown in [10] that the relation

$$1 + i\, \delta V = 1 + \varepsilon^\mu\, \mathfrak{p}_\mu + (i/2) \omega^{\varrho\sigma}\, \mathfrak{M}_{\varrho\sigma} \tag{2.19}$$

with the infinitesimal functional generators

$$\begin{aligned} \mathfrak{p}_\mu &:= \int j_a(x) P_\mu(x)\, \partial^\alpha(x)\, d^4x , \\ \mathfrak{M}_{\varrho\sigma} &:= \int j_a(x) . M_{\varrho\sigma}(x)\, \partial^\alpha(x)\, d^4x , \end{aligned} \tag{2.20}$$

and the generators $P_\mu(x)$, $M_{\varrho\sigma}(x)$ in function space holds. Substitution of (2.4) into (2.19) (2.20) gives

$$(1 + i\, \delta V) = 1 + i \sum_{k,l} a_k^+ a_l \int f_{ak}(x) [\varepsilon^\mu P_\mu(x) + \tfrac{1}{2} \omega^{\varrho\sigma} M_{\varrho\sigma}(x)]\, g_l^\alpha(x)\, d^4x . \tag{2.21}$$

On the other hand we denote the infinitesimal transformation in function space by $(1+i\,\delta\,C)$ resp. $(1+i\,\delta\,\tilde{C})$. Their elements are given by

$$(1+i\,\delta\,C)_{kl}=\delta_{kl}+i\int f_{ak}(x)\,[\varepsilon^{\mu}P_{\mu}(x)+\tfrac{1}{2}\,\omega^{\varrho\sigma}M_{\varrho\sigma}(x)]\,g_{l}^{\alpha}(x)\,\mathrm{d}^4x = (1+i\,\delta\,\tilde{C})_{lk}\,. \tag{2.22}$$

Substitution of (2.21) (2.22) into (2.15) and direct calculation shows that (2.15) is an identity up to first order terms, q. e. d.

Summarizing these results we can give the

Stat. 2.4: If an algebra of operators $\{a_k^+, a_k\}$ satisfying the relations (2.5) (2.6) (2.13) is given and if additionally a function space $\{f_{ak}(x)\}$ satisfying (2.1) (2.3) exists, then a functional space satisfying the conditions (1.12) to (1.20) can be constructed explicitly.

Proof: The proof makes use of the statements 2.1 – 2.3 q. e. d.

From statement 2.4 an essential conclusion for the construction of functional spaces can be drawn. Obviously by the expansion (2.4) a separation of the transformation properties and the algebraic properties can be achieved. Therefore we can look on the one hand for a realization of an algebra without transformation properties and on the other hand for a function space without algebraic requirements but proper transformation properties. This point of view has been already emphasized by Stumpf [6]. Therefore the problem of constructing appropriate functional spaces can be subdivided into two independent problems of algebraic and analytic nature. The algebraic problem can be solved immediatly. As is well known, a set of operators $\{a_k^+, a_k, 1\leqq k<\infty\}$ can be constructed explicitly satisfying (2.5) and (2.6). The explicit representation of such a Jordan Wigner algebra has been given for example in [11]. Also a groundstate $|\,\varphi_0\rangle$ can be constructed explicitly in the framework of this algebra. So no more interest is given in this problem. Therefore for the construction of appropriate functional spaces we have only to look for appropriate function spaces which satisfy (2.1) (2.3). Due to the construction of the Jordan Wigner algebra it is essential that only such function spaces are used which represent separable Hilbert spaces, i.e. which have a denumerable set of indices. It will be seen that this condition together with conditions (2.1) (2.3) are the nontrivial content of the constructive approach to spinorial functional spaces.

3. On Mass Shell Decomposition

According to the results of Sect. 2 we have only to look for an appropriate system of spinorial base functions which is complete, denumerable and satisfies (2.1) and (2.3). To achieve this we consider the Fourier decomposition of an arbitrary spinorial function $\chi_a(x)$ in Minkowski space. For convenience we consider an ordinary spinor with the transformation property

$$\chi_a{}'(x')=S_a{}^{\beta}(\Lambda)\,\chi_{\beta}\,(\Lambda^{-1}(x'-d))\,. \tag{3.1}$$

The transition to real spinorial base functions can be achieved later by the transformation (1.1) (1.2). Especially the calculations of section 2 are done with such real spinorial functions.

The Fouriertransform of $\chi_a(x)$ reads

$$\chi_a(x)=\int e^{-ipx}\,\tilde{\chi}_a(p)\,\mathrm{d}^4p\,. \tag{3.2}$$

The integral in (3.2) covers the entire Minkowski space. It allows the Poincaré-invariant decomposition

$$\chi_a(x)=\int_{L_t} e^{-ipx}\,\tilde{\chi}_a(p)\,\mathrm{d}^4p+\int_{L_s} e^{-ipx}\,\tilde{\chi}_a(p)\,\mathrm{d}^4p=\chi_a{}^{\mathrm{t}}(x)+\chi_a{}^{\mathrm{s}}(x) \tag{3.3}$$

where L_{t} is the timelike domain with $p^2\geqq 0$, while L_{s} is the spacelike domain with $p^2<0$. The integral (3.3) can be further decomposed. In L_{t} we have

$$\begin{aligned}\chi_a{}^{\mathrm{t}}(x)&=\int_0^{\infty}\mathrm{d}m^2\int_{-\infty}^{\infty}\mathrm{d}^4p\;\tilde{\chi}_a{}^{\mathrm{t}}(p)\,\delta(p^2-m^2)\,e^{-ipx}\\&=\int_0^{\infty}\mathrm{d}m^2\int_{-\infty}^{\infty}\mathrm{d}^3p\;2\,(\mathfrak{p}^2+m^2)^{-1/2}\,e^{-ipx}\,\chi_a{}^{\mathrm{t}}(\mathfrak{p},m)+\int_0^{\infty}\mathrm{d}m^2\int_{-\infty}^{\infty}\mathrm{d}^3p\;2\,(\mathfrak{p}^2+m^2)^{-1/2}\,e^{ipx}\,\chi_a{}^{-}(\mathfrak{p}\,m)\,.\end{aligned} \tag{3.4}$$

In L_{s} we have

$$\chi_a{}^{\mathrm{s}}(x)=\int_0^{\infty}\mathrm{d}m^2\int_{-\infty}^{\infty}\mathrm{d}^4p\;\tilde{\chi}_a{}^{\mathrm{s}}(p)\,\delta(p^2+m^2)\,e^{-ipx}\,. \tag{3.5}$$

Introducing the variables ϑ, φ, m by

$$p_1 = (m^2 + p_0^2)^{1/2} \cos\vartheta \sin\varphi , \quad p_2 = (m^2 + p_0^2)^{1/2} \cos\varphi \sin\vartheta , \quad p_3 = (m^2 + p_0^2)^{1/2} \sin\vartheta , \tag{3.6}$$

and carrying out the $|\mathfrak{p}|$-integration one obtains

$$\chi_\alpha^s(x) = \int_{-\infty}^{\infty} \mathrm{d}p_0 \int_0^{\infty} \mathrm{d}m^2 \tfrac{1}{2}(p_0^2 + m^2)^{1/2} \int \mathrm{d}\Omega\, e^{-ipx} \tilde{\chi}_\alpha^s(\mathfrak{w}, p_0, m) \tag{3.7}$$

$$= \int_0^{\infty} \mathrm{d}p_0 \int_0^{\infty} \mathrm{d}m^2 \tfrac{1}{2}(p_0^2 + m^2)^{1/2} \int \mathrm{d}\Omega\, e^{-ipx} \tilde{\chi}_\alpha^s(\mathfrak{w}, p_0, m) + \int_0^{\infty} \mathrm{d}p_0 \int_0^{\infty} \mathrm{d}m^2 \tfrac{1}{2}(p_0^2 + m^2)^{1/2} \int \mathrm{d}\Omega\, e^{ipx} \tilde{\chi}_\alpha^s(\mathfrak{w}, p_0, m)$$

with $\mathfrak{w}$ defined by $\mathfrak{p} = (p_0^2 + m)^{1/2}\, \mathfrak{w} = |\mathfrak{p}|\, \mathfrak{w}$.

It has to be noted that the splitting of the integral in (3.7), e.g. the separation into positive and negative energy parts is not Poincaré invariant as the energy sign is not invariant in the spacelike region.

(3.4) and (3.7) show that it is not possible to represent a general spinorial wave function by a set of functions which are defined on the mass shell, if one admits only real but not imaginary masses too, as in L_s, $m^2 = p^2 < 0$ for all p^2 is valid. Additionally it is required that this on mass shell decomposition follows the proper transformation law (3.1) for spinorial wave functions. To show that this requirement can be satisfied, we have to perform some grouptheoretical considerations.

As is well known, the behaviour of spinorial wave functions is governed by the Dirac equation. Its solutions for a fixed mass m span a representation space of an irreducible representation with mass m, $m^2 > 0$ and spin $1/2$. Assuming such Dirac spinors to be used in the decomposition (3.4) the integration over m in (3.4) destroys irreducibility but does not affect the transformation law (3.1) as $S(\Lambda)$ does not depend on m [12, 13]. Therefore for the decomposition of the timelike part (3.4) ordinary Dirac spinors for the various masses m can be used.

To realize the spacelike decomposition (3.7) by appropriate Dirac spinors one can try to replace in the Dirac equation m by $i\,m$. The solutions of such an equation span for fixed m a representation space of a nonunitary representation of the Poincaré group with $m^2 = p^2 < 0$ and $s = 1/2$ [14]. That the transformation law (3.1) for the case $m^2 > 0$ with $S(\Lambda)$ finite dimensional defines a unitary representation depends on the unitarity representation of the finite dimensional representation of the corresponding little group which is SU_2 for $m^2 > 0$. For $m^2 < 0$ the little group is $SU(1, 1)$ which is noncompact and therefore has no finite dimensional unitary representations. So the transformation law (3.1) with $S(\Lambda)$ finite dimensional defines necessarily a nonunitary representation of the Poincaré group for $m^2 < 0$. This has the consequence that the underlying representation space has no invariant bilinear form which is positive definite, so that we have to deal with an indefinite metric space.

Dirac equations with imaginary masses have been considered by Tanaka [15] who investigated the invariant functions and quantized the theory to investigate tachyons with spin. Recently Bandukwala and Shay [16] proposed a theory of spin $1/2$-tachyons similar to Tanaka's. As stated by the authors, their results allow different interpretations which are quite contradictious. For the approach presented here it is not necessary to discuss these contradictions. It suffices to refer to Tanaka's theory.

As observed by Tanaka, the general spinor equation which is invariant under proper Lorentz transformations and under Poincaré transformations (3.1) has the form

$$(\gamma_\mu\, \partial^\mu + \alpha + \beta\, \gamma^5)\, \varphi(x) = 0 \tag{3.8}$$

with α and β complex numbers: $\alpha = \alpha_1 + i\,\alpha_2$, $\beta = \beta_1 + i\,\beta_2$, and $\gamma^5 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3$, $(\gamma^5)^2 = -1$. Imposing different restrictions on α_1, α_2, β_1, β_2 and applying a suitable transformation from (3.8) two essentially different equations can be derived. In the first case one obtains the common Dirac equation with

$$(i\,\gamma_\mu\, \partial^\mu - m)\, \varphi(x) = 0 \tag{3.9}$$

with $m = (\alpha_1^1 + \alpha_2^2)^{1/2}$ while in the second case one obtains the two equivalent equations

$$(\gamma_\mu\, \partial^\mu - m)\, \varphi(x) = 0 \tag{3.10 a}$$

resp.

$$(\gamma_\mu\, \partial^\mu - \gamma^5 m)\, \varphi(x) = 0\,. \tag{3.10 b}$$

The solutions of (3.10 a) (3.10 b) satisfy also the Klein Gordon equation for imaginary masses.

As Eqs. (3.9) (3.10) are special cases of (3.8) their spinorial wave functions obey the same transformation law which leaves (3.8) invariant, namely the transformations (3.1). Therefore we can use the solutions of (3.9) (3.10) for various m to construct a complete base system defined on the entire Minkowski space and obeying the transformation law (3.1). First we treat the case p timelike. Making for the solutions of (3.9) the usual plane wave ansatz

$$\varphi_\beta(x) = e^{-ipx}\, u_\beta(p) + e^{ipx}\, v_\beta(p) \qquad (3.11)$$

we obtain a Poincaré invariant splitting into positive and negative energy solutions. u and v obey the following orthonormality and completeness relations:

$$\bar u^r(p) u^s(p) = -\bar v^r(p) v^s(p) = \delta^{rs}, \quad r,s = 1,2,$$
$$\sum_r \bar u_\alpha{}^r(p) u_\beta{}^r(p) - \sum_r \bar v_\alpha{}^r(p) v_\beta{}^r(p) = \delta_{\alpha\beta}. \qquad (3.12)$$

We now expand $\tilde\chi_\alpha{}^+(p)$ with respect to $u_\alpha{}^r(p)$ and $\tilde\chi_\alpha{}^-(p)$ with respect to $v_\alpha{}^r(p)$ and obtain

$$\chi_\alpha{}^{\mathrm t}(x) = \int_0^\infty \mathrm dm^2 \int \mathrm d^3p\, \tfrac12 (\mathfrak p^2+m^2)^{-1/2}\, e^{-ipx} \sum_r \tilde\chi_r{}^+(p) u_\alpha{}^r(p)$$
$$+ \int_0^\infty \mathrm dm^2 \int \mathrm d^3p\, \tfrac12 (\mathfrak p^2+m^2)^{-1/2}\, e^{ipx} \sum_r \tilde\chi_r{}^-(p) v_\alpha{}^r(p). \qquad (3.13)$$

Making the substitution $y = (\mathfrak p^2+m^2)^{1/2} - |\mathfrak p|$ we obtain

$$\chi_\alpha{}^{\mathrm t}(x) = \int_0^\infty \mathrm dy \int \mathrm d^3p \exp\{-i(y+|\mathfrak p|)x_0 + i\,\mathfrak p\,\mathfrak x\} \sum_r \tilde\chi_r{}^+(\mathfrak p,y) u_\alpha{}^r(\mathfrak p,y)$$
$$+ \int_0^\infty \mathrm dy \int \mathrm d^3p \exp\{i(y+|\mathfrak p|)x_0 - i\,\mathfrak p\,\mathfrak x\} \sum_r \tilde\chi_r{}^-(\mathfrak p,y) v_\alpha{}^r(\mathfrak p,y) =: \chi_\alpha{}^{\mathrm t+}(x) = \chi_\alpha{}^{\mathrm t-}(x). \qquad (3.14)$$

We now expand $\tilde\chi_r{}^+(\mathfrak p,y)$ and $\tilde\chi_r{}^-(\mathfrak p,y)$ into someone fourdimensional complete system $\{h_\nu(\mathfrak p,y), 1 \leqq \nu < \infty\}$ satisfying the orthonormality conditions

$$\int_0^\infty \mathrm dy \int_{-\infty}^\infty \mathrm d^3p\, h_\nu{}^\times(\mathfrak p,y)\, h_{\nu'}(\mathfrak p,y) = \delta_{\nu\nu'} \qquad (3.15)$$

and obtain for $\chi_\alpha^{\mathrm t+}(x)$ resp. $\chi_\alpha^{\mathrm t-}(x)$ the following series

$$\chi_\alpha{}^{\mathrm t+}(x) = \int_0^\infty \mathrm dy \int_{-\infty}^\infty \mathrm d^3p \sum_{r,\nu} \chi_{r\nu}^+ h_\nu(\mathfrak p,y) u_\alpha{}^r(\mathfrak p,y) \exp\{-i(y+|\mathfrak p|)x_0 + i\,\mathfrak p\,\mathfrak x\} =: \sum_{r,\nu} \chi_{r\nu}^+ u_\alpha^{r\nu}(x) \qquad (3.16)$$

resp.

$$\chi_\alpha{}^{\mathrm t-} = \int_0^\infty \mathrm dy \int_{-\infty}^\infty \mathrm d^3p \sum_{r,\nu} \chi_{r\nu}^- h_\nu(\mathfrak p,y) v_\alpha{}^r(\mathfrak p,y) \exp\{i(y+|\mathfrak p|)x_0 - i\,\mathfrak p\,\mathfrak x\} =: \sum_{r,\nu} \chi_{r\nu}^- v_\alpha^{r\nu}(x). \qquad (3.17)$$

Then it can be easily shown that the following relations hold

$$\int \bar u_\alpha^{r\nu}(x) u_\alpha^{r'\nu'}(x)\, \mathrm d^4x = \delta_{\nu\nu'}\,\delta_{rr'}, \qquad (3.18\,\mathrm a)$$
$$\int \bar v_\alpha^{r\nu}(x) v_\alpha^{r'\nu'}(x)\, \mathrm d^4x = -\delta_{rr'}\,\delta_{\nu\nu'}, \qquad (3.18\,\mathrm b)$$
$$\int \bar v_\alpha^{r\nu}(x) u_\alpha^{r'\nu'}(x)\, \mathrm d^4x = \int \bar u_\alpha^{r\nu}(x) v_\alpha^{r'\nu'}(x)\, \mathrm d^4x = 0. \qquad (3.18\,\mathrm c)$$

Calculating therefore the scalar product between two general spinorial wave functions $\chi_\alpha{}^+(x)$ and $\xi_\alpha{}^+(x)$ we therefore obtain by means of the preceding formulas

$$\int \bar\chi_\alpha{}^{\mathrm t}(x) \xi_\alpha{}^{\mathrm t}(x)\, \mathrm d^4x$$
$$= \int \mathrm d^4x \sum_{r,\nu} (\chi_{r\nu}^- \bar v_\alpha^{r\nu}(x) + \chi_{r\nu}^+ \bar u_\alpha^{r\nu}(x))$$
$$\times \sum_{r,\nu} (\xi_{r'\nu'}^- v^{r'\nu'}(x) + \xi_{r'\nu'}^+ v^{r'\nu'}(x))$$
$$= \sum_{r,\nu} (-\chi_{r\nu}^- \xi_{r\nu}^- + \chi_{r\nu}^+ \xi_{r\nu}^+). \qquad (3.19)$$

(3.19) is not positive definite due to the normalization of v according to (3.18 b). Therefore we define a new scalar product by the prescription

$$\langle v^{r\nu}, v^{r'\nu'}\rangle := -\int \bar v_\alpha^{r\nu}(x) v_\alpha^{r'\nu'}(x)\, \mathrm d^4x = \delta_{rr'}\,\delta_{\nu\nu'} \qquad (3.20)$$

while (3.18 a) (3.18 c) remain unchanged. With (3.20) we get now for the scalar product of $\chi_\alpha{}^{\mathrm t}(x)$ and $\xi_\alpha{}^{\mathrm t}(x)$ the expression

$$\langle \chi_\alpha{}^{\mathrm t}(x), \xi_\alpha{}^{\mathrm t}(x)\rangle = \sum_{r,\nu} (\chi_{r\nu}^- \xi_{r\nu}^- + \chi_{r\nu}^+ \xi_{r\nu}^+). \qquad (3.21)$$

The redefinition of the scalar product is allowed because of the relation (3.18 c) and the invariance of the splitting into positive and negative energy parts. Therefore the change of sign in the scalar product is not affected by Poincaré transformations. We stress that this statement only holds for the case timelike.

To treat the spacelike case we take as a starting point the Eqs. (3.10 a, b). Multiplying (3.10 b) with $-\gamma^5$ from the left we get the equation

$$(\gamma_\mu{}' \, \partial^\mu - m)\varphi(x) = (\gamma_5 \, \gamma_\mu \, \partial^\mu - m)\varphi(x) = 0 . \qquad (3.22)$$

It is easily shown that the set $\gamma_\mu{}' = \gamma_5 \, \gamma_\mu$ has all required properties of γ-matrices, so that equation (3.22) resp. (3.10 a) follows from (3.10 b). On the other hand, starting from Eq. (3.10 a), one observes that every set of γ-matrices can be written as

$$\gamma_\mu = \gamma_5{}' \, \gamma_\mu{}' \qquad (3.23)$$

with $\gamma_\mu{}'$ another set of Diracmatrices and $\gamma_5{}' = \gamma_5$. Multiplying with $\gamma_5{}'$ from the left yields (3.10 b).

Therefore, equations (3.10 a) and (3.10 b) are equivalent, as stated above. The nonsingular matrix which connects the two sets of γ-matrices by the relation

$$\gamma_\mu{}' = S_{\gamma\mu} \, S^{-1} \qquad (3.24)$$

is given by $S = (1 + \gamma_5)$.

We are now in the strange situation that we have two equivalent equations with obvious different transformation properties under the full Lorentz group. Equation (3.10 a) is perfectly invariant, but the γ_5-term in (3.10 b) may change sign if reflections are involved, according to the relation

$$S^{-1}(\Lambda) \, \gamma_5 \, S(\Lambda) = \gamma_5 \det \Lambda . \qquad (3.25)$$

As is well known, the spinor representations $S(\Lambda)$ can be expressed by γ-matrices. Now, starting with (3.10 a) which is invariant under the full group and multiplying with γ_5, one gets (3.10 b) in terms of $\gamma_\mu{}'$. Evidently, this equation is invariant, too, under the full group, if one uses the same $S(\Lambda)$ as for (3.10 b).

One sees that it is a matter of convenience to regard (3.10 a, b) as parity violating or not. It is possible to redefine the parity operation for (3.10 b) making it a perfect invariant equation.

Looking for the invariant bilinear form, we start from (3.10 b)

$$(\gamma_\mu \, \partial^\mu - \gamma_5 \, m)\varphi(x) = 0 . \qquad (3.10\,\text{a})$$

The conjugate equation reads

$$\varphi^+(x) \, (\gamma_\mu{}^+ \, \partial^\mu - \gamma_5{}^+ \, m) = \overline{\varphi}(x) \, (\gamma_\mu \, \partial^\mu - \gamma_5 \, m) = 0 . \qquad (3.26)$$

Multiplying from the right with γ_5 and using $[\gamma_\mu \gamma_5]_+ = 0$ we get

$$\hat{\varphi}(x) \, (\gamma_\mu \, \partial^\mu + \gamma_5 \, m) = 0 \qquad (3.27)$$

with

$$\hat{\varphi}(x) := \varphi^+(x) \, \gamma_0 \, \gamma_5 . \qquad (3.28)$$

By (3.27) and (3.10 b) one can show that the current $j_\mu := \hat{\varphi} \, \gamma_\mu \, \varphi$ is conserved:

$$\partial_\mu \, j^\mu(x) = \partial_\mu \, \hat{\varphi}(x) \gamma^\mu \varphi(x) = 0 . \qquad (3.29)$$

Therefore, we have to deal with $\hat{\varphi}\varphi$ in defining scalar products rather than with $\overline{\varphi}\varphi$ which can be shown to vanish.

For actual calculations, we proceed with (3.10 a) which is easier to handle. Making the planewave ansatz

$$\varphi_\beta(x) = w_\beta(\beta) \, e^{-ipx} + z_\beta(p) \, e^{ipx} \qquad (3.30)$$

we obtain

$$(-i \, \gamma_\mu \, p^\mu - m) \, w(p) \, e^{-ipx} + (i \, \gamma_\mu \, p^\mu - m) \, z(p) \, e^{ipx} = 0 . \qquad (3.31)$$

In (3.31) the two terms cannot be treated separately, because of the possible change of the energysign under Poincaré transformations for spacelike momenta. But because of their linear independence we can equal to zero each term of (3.31) separately. Then by introducing an additional variable ε, which takes the values $\varepsilon = \pm 1$ we can rewrite (3.31) in the following form

$$(i \, \gamma_\mu \, p^\mu - \varepsilon \, m) \, w^\varepsilon(p) = 0 \qquad (3.32)$$

using the definitions $w^+ := w$ and $w^- := z$.

Going to the rest frame $(0, 0, 0, m)$ we get

$$\gamma_3 \, w^\varepsilon = \varepsilon \, i \, w^\varepsilon \qquad (3.33)$$

with the solutions

$$w_+^+ = (2)^{-1/2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad w_-^+ = (2)^{-1/2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}, \quad w_+^- = (2)^{-1/2} \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix}, \quad w_-^- = (2)^{-1/2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}. \qquad (3.34)$$

The solutions (3.34) are normalized according to

$$\langle \hat{w}, w \rangle := \hat{w}_\sigma^\varepsilon \, w_{\sigma'}^\varepsilon = \varepsilon \, \sigma \, \delta_{\sigma\sigma'} \, \delta_{\varepsilon\varepsilon'} , \quad \varepsilon, \sigma = \pm 1 \qquad (3.35)$$

and fulfill the completeness relation

$$\sum_{\sigma, \varepsilon} \varepsilon \, \sigma \, \hat{w}_{\alpha\sigma}^\varepsilon \, w_{\alpha'\sigma}^\varepsilon = \delta_{\alpha\alpha'} . \qquad (3.36)$$

The transformation

$$S := i\,(2\,m\,(p_3 - m))^{-1/2}\,(\gamma_\mu\, p^\mu - \gamma_3\, m) \quad (3.37)$$

leaves the scalar product (3.35) invariant

$$\langle S\,\hat{w}, S\,w\rangle = \langle \hat{w}, w\rangle\,. \quad (3.38)$$

Application of S to $w_\sigma{}^\varepsilon$ gives with (3.33)

$$w_\sigma{}^\varepsilon(p) = (2\,m\,(p_3 - w))^{-1/2}\,(i\,\gamma_\mu\, p^\mu + \varepsilon\,\gamma_5\, m)\, w_\sigma{}^\varepsilon\,. \quad (3.39)$$

$w_\sigma{}^\varepsilon(p)$ solves automatically (3.32) and is therefore the solution in a general frame of reference. Consequently, S is the representation matrix of a boost for p spacelike.

We now proceed analogously to the timelike case. We expand in (3.7) $\chi_\alpha^{s-}(\Omega, p_0, m)$ and $\chi_\alpha^{s+}(\Omega, p_0, m)$ into the sets $w_{\alpha,\sigma}^-(p)$ and $w_{\alpha,\sigma}^+(p)$ with respect to α and into any suitable normalized set with respect to (Ω, p_0, m). Then one obtains analogously to the timelike case an expansion of $\chi_\alpha{}^s(x)$ in the form

$$\chi_\alpha^s(x) = \chi_\alpha^{s+}(x) + \chi_\alpha^{s-}(x) = \sum_{\sigma,\nu}(\chi_{\sigma,\nu}^+\, w_\alpha^{+\sigma\nu}(x) + \chi_{\sigma\nu}^-\, w_\alpha^{-\sigma\nu}(x)) \quad (3.40)$$

where the expansion functions are normalized according to

$$\int \hat{w}_\alpha^{+\sigma}(x)\, w_\alpha^{+\sigma'}(x)\,\mathrm{d}^4x = \sigma\,\delta_{\sigma\sigma'}\,, \quad (3.41\text{ a})$$

$$\int \hat{w}_\alpha^{-\sigma}(x)\, w_\alpha^{-\sigma'}(x)\,\mathrm{d}^4x = -\sigma\,\delta_{\sigma\sigma'}\,, \quad (3.41\text{ b})$$

$$\int \hat{w}_\alpha^{+\sigma}(x)\, w_\alpha^{-\sigma'}(x)\,\mathrm{d}^4x = \int \hat{w}_\alpha^{-\sigma}(x)\, w_\alpha^{+\sigma'}(x)\,\mathrm{d}^4x = 0\,. \quad (3.41\text{ c})$$

Different from the timelike case, it is not possible to get rid of the sign in (3.41) in an invariant manner by a mere change of definition, as the energy sign and the spin projection are not invariant and therefore w^+ and w^- may interchange. So we get indefinite metric in the underlying space build up by the w^+ and w^-, reflecting the fact that the appertaining representations of the Poincaré group are not unitary.

Summarizing the results of this investigation, we can state that it is possible to construct a complete, linearly independent discrete set

$$\{f_\alpha^{\sigma\nu}(x), 1 \leqq \nu < \infty\}: = \{u_\alpha^{\sigma\nu}(x), v_\alpha^{\sigma\nu}(x), w_\alpha^{\sigma\nu}(x), z_\alpha^{\sigma\nu}(x), 1 \leqq \nu < \infty\} \quad (3.42)$$

which covers the entire Minkowski space with $w_\alpha^{\sigma\nu} \equiv w_\alpha^{+\sigma\nu}$ and $z_\alpha^{\sigma\nu} \equiv w_\alpha^{-\sigma\nu}$. Observing that u, v and w, z are defined on different supports, they are orthogonal. Combining this with the orthonormality relations (3.18) (3.41) a dual set $\{g_\alpha^{\sigma\nu}(x)\}$ can be defined by

$$\{g_\alpha^{\sigma\nu}(x), 1 \leqq \nu < \infty\}: \{\bar{u}_\alpha^{\sigma\nu}, -\bar{v}_\alpha^{\sigma\nu}, \sigma\,\hat{w}_\alpha^{\sigma\nu}, -\sigma\,\hat{z}_\alpha^{\sigma\nu}, 1 \leqq \nu < \infty\}\,. \quad (3.43)$$

Additionally, any element of the set exhibits the right transformation properties and the scalar product of the sets (3.42) (3.43) is invariant.

4. Functional Scalar Products

Nothing has been said so far about the metric of the functional space constructed in section 1. Like in ordinary Minkowski space the scalar products (1.19) between the base elements of the original and the dual set give no hint on the metrical structure of the space in consideration. For this purpose, the scalar products between the base elements of the original set themselves have to be considered, i.e. the metrical fundamental tensor of the functional space has to be derived. To stress the analogy to ordinary Minkowski space we write the general nonlinear spinor field functional state (1.11) in the symbolic form

$$|\mathfrak{T}(j, a)\rangle = \sum_n \tau^n(a)\,|D_n\rangle \quad (4.1)$$

where $\tau^n(a)$ has to be identified with (1.9). Defining then

$$\langle\mathfrak{T}(j, b)| := (|\mathfrak{T}(j, b)\rangle)^+ = \sum_n (|D_n\rangle)^\times\,\tau^n(b)^\times \quad (4.2)$$

with $\langle D_n| := (|D_n\rangle)^+$ a scalar product can be defined formally by

$$\langle\mathfrak{T}(j, b)|\mathfrak{T}(j, a)\rangle = \sum_{n,m} \tau^n(b)^\times\, g_{nm}\,\tau^m(a) \quad (4.3)$$

with

$$g_{nm} := \langle D_n | D_m\rangle \quad (4.4)$$

For spaces with indefinite metric we have $\langle D_n| \neq \langle D^n|$ i.e. the Hermitean conjugate base vector $\langle D_n|$ is unequal to the dual base vector $\langle D^n|$. The formal scalar product (4.3) can be written equivalently by means of the weigthing functional

$$\mathfrak{G}(j) := \sum_{n,m} |D^n\rangle\, g_{nm}\,\langle D^m|\,. \quad (4.5)$$

Observing $\langle D^i | D_j\rangle = \delta_j{}^i$ according to (1.19) we obtain

$$\langle\mathfrak{T}(j, b)|\mathfrak{T}(j, a)\rangle = \langle\mathfrak{T}(j, b)|\mathfrak{G}(j)|\mathfrak{T}(j, a)\rangle\,. \quad (4.6)$$

Therefore, for the formation of scalar products it is essential to calculate (4.5). This can be done by using the expansion (2.9). Then by (4.2) follows

$$\langle D_n(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n})| = \sum_{k_1\dots k_n} f^+_{\alpha_1 k_1}(x_1)\dots f^+_{\alpha_n k_n}(x_n)\,(|D_n(k_1\dots k_n)\rangle)^+ . \tag{4.7}$$

But from (2.10) and (2.12) immediately follows

$$(|D_n(k_1\dots k_n)\rangle)^+ = \langle D_n(k_1\dots k_n)| = \langle D^n(k_1\dots k_n)| . \tag{4.8}$$

Observing this and applying (2.14) we obtain

$$\langle D_n(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n})|D_m(\underset{\beta_1}{x'_1}\dots\underset{\beta_m}{x'_m})\rangle = \delta_{nm}\sum_{k_1\dots k_n} p \sum_{\lambda_1\dots\lambda_n}(-1)^p f^+_{\underset{\alpha_1}{k_1}}(x_1) f_{\underset{\beta_{\lambda_1}}{k_1}}(x'_{\lambda_1})\dots f^+_{\underset{\alpha_n}{k_n}}(x_n) f_{\underset{\beta_{\lambda_1}}{k_n}}(x'_{\lambda_n}) \tag{4.9}$$

Therefore, by defining

$$g_{nm}(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n}, \underset{\beta_1}{x'_1}\dots\underset{\beta_m}{x'_m}) := \delta_{nm}\sum_{k_1\dots k_n} p \sum_{\lambda_1\dots\lambda_n}(-1)^p f^+_{\underset{\alpha_1}{k_1}}(x_1) f_{\underset{\beta_{\lambda_1}}{k_1}}(x'_{\lambda_1})\dots f^+_{\underset{\alpha_n}{k_n}}(x_n) f_{\underset{\beta_{\lambda_n}}{k_n}}(x'_{\lambda_n}) \tag{4.10}$$

the metrical fundamental tensor in functional space can be written

$$\mathfrak{G}(j) = \sum_{nm}\int |D^n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n})\rangle\, g_{nm}(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n}, \underset{\beta_1}{x'_1}\dots\underset{\beta_m}{x'_m})\,\langle D^m(\overset{\beta_1}{x'_1}\dots\overset{\beta_m}{x'_m})|\,\mathrm{d}x\,\mathrm{d}x' . \tag{4.11}$$

For the explicit numerical evaluation and application of (4.11) it is convenient to use a representation where $g_{nm}(x, x')$ is diagonal with respect to the space coordinates. To achieve this we perform a Fouriertransformation to (4.11) leading to

$$\mathfrak{G}(j) := \sum_{nm}\int |D^n(\overset{\alpha_1}{p_1}\dots\overset{\alpha_n}{p_n})\rangle\, \tilde{g}_{nm}(\underset{\alpha_1}{p_1}\dots\underset{\alpha_n}{p_n}, \underset{\beta_1}{p'_1}\dots\underset{\beta_m}{p'_m})\,\langle D_m(\overset{\beta_1}{p'_1}\dots\overset{\beta_m}{p'_m})|\,\mathrm{d}p\,\mathrm{d}p' . \tag{4.12}$$

As g_{nm} resp $\tilde{g}_{nm}$ is the direct product of

$$\tilde{g}_{11}(p_\alpha p'_\beta) := \sum_k \tilde{f}^+_{k\alpha}(p)\tilde{f}_{k\alpha'}(p') \tag{4.13}$$

it is sufficient to study $\tilde{g}_{11}$. Having proven the existence of the base functionals, we are allowed to use for the calculation of (4.13) any representation $\{f_{k\alpha}(p)\}$ which is useful for our purpose. Therefore, in this case we use a plane wave representation namely

$$\tilde{f}_{k\alpha} \equiv \tilde{f}_{\nu\sigma\varkappa\alpha}(p) = \delta(p - p_\nu) f_{\sigma\varkappa\alpha}(p_\nu) \tag{4.14}$$

with

$$\begin{aligned}\{f_{\sigma\varkappa\alpha}(p_\nu)\} &:= \{f_{\sigma 1\alpha}(p_\nu) = u_{\sigma\alpha}(p_\nu),\quad f_{\sigma 2\alpha}(p_\nu) = v_{\sigma\alpha}(p_\nu)\},\\ \{f^+_{\sigma\varkappa\alpha}(p_\nu)\} &:= \{f^+_{\sigma 1\alpha}(p_\nu) = \bar{u}_{\sigma\alpha}(p_\nu),\quad f^+_{\sigma 2\alpha}(p_\nu) = \bar{v}_{\sigma\alpha}(p_\nu)\}\end{aligned} \tag{4.15}$$

for p_ν timelike and

$$\begin{aligned}\{f_{\sigma\varkappa\alpha}(p_\nu)\} &:= \{f_{\sigma 1\alpha}(p_\nu) = w_{\sigma\alpha}(p_\nu),\quad f_{\sigma 2\alpha}(p_\nu) = z_{\sigma\alpha}(p_\nu)\},\\ \{f^+_{\sigma\varkappa\alpha}(p_\nu)\} &:= \{f^+_{\sigma 1\alpha}(p_\nu) = \hat{w}_{\sigma\alpha}(p_\nu),\quad f^+_{\sigma 2\alpha}(p_\nu) = \hat{z}_{\sigma\alpha}(p_\nu)\}\end{aligned} \tag{4.16}$$

for p_ν spacelike. Then we get from (4.13):

$$\begin{aligned}\tilde{g}_{11}(p_\alpha\, p'_\beta) = &\sum_{p_\nu\in L_t}\delta(p - p_\nu)\,\delta(p' - p_\nu)\Big[\sum_\sigma\big(\bar{u}_{\sigma\alpha}(p_\nu)u_{\sigma\beta}(p_\nu) + \bar{v}_{\sigma\alpha}(p_\nu)v_{\sigma\beta}(p_\nu)\big)\Big]\\ &+ \sum_{p_\nu = L_s}\delta(p - p_\nu)\,\delta(p' - p_\nu)\Big[\sum_\sigma\big(\hat{w}_{\sigma\alpha}(p_\nu)w_{\sigma\beta}(p_\nu) + \hat{z}_{\sigma\alpha}(p_\nu)z_{\sigma\beta}(p_\nu)\big)\Big] .\end{aligned} \tag{4.17}$$

As is easily verified, the following relations hold:

$$\sum_\sigma\big(\bar{u}_{\sigma\alpha}(p_\nu)u_{\sigma\beta}(p_\nu) + \bar{v}_{\sigma\alpha}(p_\nu)v_{\sigma\beta}(p_\nu)\big) = \gamma^0_{\alpha\beta},\quad \sum_\sigma\big(\hat{w}_{\sigma\alpha}(p_\nu)w_{\sigma\beta}(p_\nu) + \hat{z}_{\sigma\alpha}(p_\nu)z_{\sigma\beta}(p_\nu)\big) = (\gamma^5\gamma_0)_{\alpha\beta} \tag{4.18}$$

for all p_ν occuring in (4.17). Therefore, in this representation (4.17) can be written

$$\tilde{g}_{11}(p_\alpha p'_\beta) = \sum_{p_\nu\in L_t}\delta(p - p')\,\delta(p' - p_\nu)\gamma_0 + \sum_{p_\nu\in L_s}\delta(p - p')\,\delta(p' - p_\nu)\gamma^5\gamma^0 \tag{4.19}$$

i.e. $\tilde{g}_{11}(p, p')$ is diagonal with respect to the space coordinates.

Introducing the notation $L_t =: L_1$, $L_s =: L_2$ and $\gamma(1) := \gamma_0$, $\gamma(2) := \gamma^5 \gamma_0$ we obtain for $\mathfrak{G}(j)$

$$\mathfrak{G}(j) = \sum_n \sum_{i_1 \dots i_n} \int_{L_{i_1}} \dots \int_{L_{i_n}} |D^n(\overset{\alpha_1}{p_1} \dots \overset{\alpha_n}{p_n})\rangle \, \gamma_{\alpha_1 \beta_1}(i_1) \dots \gamma_{\alpha_n \beta_n}(i_n) \, \langle D^n(\overset{\beta_1}{p_1} \dots \overset{\beta_m}{p_n})| \, d^4 p_1 \dots d^4 p_n \quad (4.20)$$

where the sum over $i_1 \dots i_n$ means the sum over all possible combinations of the numbers 1 and 2.

(4.20) clearly exhibits the indefiniteness of the metrical tensor and therefore of the spinorial functional representation space. In actual calculations, minus signs coming from the γ^0-matrices must be eliminated by changing them into plus, corresponding to a redefinition of the scalar product of the Dirac spinors in a Lorentz invariant manner for p timelike, as shown in Section 3. But we stress again that this procedure is not possible for p spacelike, so we cannot get rid of the indefiniteness coming from the $\gamma^5 \gamma^0$-matrices.

As has been already emphasized in preceding papers, the scalar product (4.6) with $\mathfrak{G}(j)$ from (4.20) of the base vectors in functional space is not sufficient. Dealing with physical functional states like (1.11) it is rather necessary to introduce a special physical scalar product in functional space, i.e. the scalar product (4.6) in general can not be used without modification. Without giving a detailed discussion we state that from a physical point of view this physical scalar product has to satisfy the following conditions:

a) it has to be forminvariant with respect to the symmetry operations of the corresponding invariance groups
b) it has to take into account the groundstate properties of the model in consideration
c) it has to orthonormalize all irreducible base representations of the symmetry groups in functional space.

The condition a) links the physical scalar product and the formal scalar product (4.6). As has been demonstrated by construction, the basic quantity of any functional space with relativistic transformation properties has to be the set $\{|D_n\rangle\}$ which leads to the forminvariant metrical fundamental tensor (4.11) resp. (4.20). Although no rigorous mathematical argument is known it seems to be plausible that any metrical fundamental tensor $\mathfrak{W}(j)$ of a physical scalar product

$$(\mathfrak{T}(j,a)|\mathfrak{T}(j,b)) := \langle \mathfrak{T}(j,a)|\mathfrak{W}(j)|\mathfrak{T}(j,b)\rangle \quad (4.21)$$

can only be a forminvariant functional of $\mathfrak{G}(j)$, i.e.

$$\mathfrak{W}(j) := \mathfrak{W}[\mathfrak{G}(j)] \quad (4.22)$$

in order to satisfy condition a). Writing $\mathfrak{G}(j) = \sum_n \mathfrak{g}_n(j)$ in preceding papers

$$\mathfrak{W}(j,\varepsilon) := \sum_n \varepsilon^n \, \mathfrak{g}_n(j) \quad (4.23)$$

with

$$(\mathfrak{T}(j,a)|\mathfrak{T}(j,b)): = \lim_{\varepsilon \to 0} {}^r\langle \mathfrak{T}(j,a,\varepsilon)|\mathfrak{W}(j,\varepsilon)|\mathfrak{T}(j,b,\varepsilon)\rangle^r \quad (4.24)$$

has been proposed [17] which is a suitable scalar product for generalized free fields [18], but which also has been applied successfully to spinor field functionals [19]. Other versions are in preparation, but shall not be discussed here [20]. Concluding this section it should be only emphasized that it is plausible to define the ghost particles in functional space. This seems to be a very promising approach to the understanding of field theories with indefinite metric.

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